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A Note on Train Algebras

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Introduction

In the twenties and thirties of this century some distinguished classes of commutative but nonassociative algebras, i.e. Bernstein algebras, train algebras, genetic algebras and special train algebras, were introcuced by some authors who had researched algebraic models in theoretical population genetics. (Particularly in 1939 Etherington, I.M.H. showed that special train algebras model many concrete biological situations, cf. Wörz－Busekros[1].) All these algebras are baric while they in general are not even power－associative and have not any unit element. A survey on the results about these algebras and thier background can be found in Wörz－Busekros[1], which will be used as a basic reference in the present paper.

Our purpose is to show some properties about the structure of train algebras(§ 2. Theorem A), which might contribute to classify train algebras of rank greater than 2, and to present a characterization of special train algebras (§ 3. Theorem B), which is an analogy with a characterization of genetic algebras given by Holgate, P. in 1972.
In § 1 we shall recall some relevant definitions and results from Wörz－Busekros[1].

1. Preliminaries

Let $\mathfrak{A}$ be an n-dimensional commutative algebra over a field K, which need neither be power-associative nor have a unit element. In $\mathfrak{A}$ the product of more than two factors is not well defined by the factors and their ordering alone. The association of the factors must be specified. But we will make no differences between left- and right-properties on products of factors in $\mathfrak{A}$ because $\mathfrak{A}$ is commutative.

We define a special type of powers by induction as following:

Let a be an element of $\mathfrak{A}$ and $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$. Then for all r in N(the natural numbers),

$$a^1:=a,\ \ a^{r+1}:=(a^r)a(=a(a^r))$$

are called the principal powers of a. Similarly the principal powers of $\mathfrak{B}$ are defined by

$$\mathfrak{B}^1:=\mathfrak{B},\ \ \mathfrak{B}^{r+1}:=\langle(\mathfrak{B}^r)\mathfrak{B}\rangle \text{ for all } r \text{ in } N,$$

where the symbol $\langle\mathfrak{S}\rangle$ means the linear span over K of all elements of $\mathfrak{S}$ for every subset $\mathfrak{S}$ of $\mathfrak{A}$.

An element a of $\mathfrak{A}$ is called nilpotent of index k if $a^k=0$ but $a^{k-1}\neq 0$ and a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is called nilpotent of index k if $\mathfrak{B}^k=\{0\}$ but $\mathfrak{B}^{k-1}\neq\{0\}$.

Now following the notation of Wörz-Busekros [1], we define the rank equation of an algebra $\mathfrak{A}$. Let $F:=K(\xi_1,\dots,\xi_n)$ be the field of rational functions in (associative and commutative) indeterminates $\xi_1,\dots,\xi_n$ over K and let $\mathfrak{A}_F:=\mathfrak{A}\otimes_K F$ be the scalar extention of $\mathfrak{A}$ to the extention field F of K.
Then if we choose a fixed basis $a_1,\dots,a_n$ in $\mathfrak{A}$, by means of the specialization $\xi_i\to\alpha_i\in K$, $i=1,\dots,n$, in the element $\underline{x}=\sum_{i=1}^{n}\xi_i a_i$ of $\mathfrak{A}_F$, we can obtain any element $a=\sum_{i=1}^{n}\alpha_i a_i$ of $\mathfrak{A}$. ($\underline{x}$ is called a generic ele-

ment of $\mathfrak{A}$.) We consider polynomials in principal powers in one commutative indeterminate X which need not be associative,

$$X^0 := 1 \in K,\ X^1 := X,\ X^{r+1} = X^r X, \text{ for all } r \text{ in } N.$$

Definition 1. Let $\underline{x} = \sum_{i=1}^{n} \xi_i a_i \in \mathfrak{A}_F$ be a generic element of $\mathfrak{A}$.
The normalized polynomial

$$q_{\underline{x}}(X) = X^r + \beta_1(\xi_1,\ldots,\xi_n)X^{r-1} + \ldots + \beta_{r-1}(\xi_1,\ldots,\xi_n)X \tag{1}$$

with $\beta_i(\xi_1,\ldots,\xi_n) \in F, i=1,\ldots,r-1$, and without absolute term which anihilates $\underline{x}$ and has minimal degree is called the rank polynomial of $\mathfrak{A}$.
The degree of $q_{\underline{x}}$ is called the rank of $\mathfrak{A}$.

It is known that each coefficient $\beta_i(\underline{x}) := \beta_i(\xi_1,\ldots,\xi_n)$ of $q_{\underline{x}}$ is homogeneous of degree i in the coefficients $\xi_1,\ldots,\xi_n$ of $\underline{x}$, $i=1,..,r-1$, (cf.[1], Theorem 2.16).

Remark. It is known that the rank polynomial $q_{\underline{x}}(X)$ of $\mathfrak{A}$ coincides either to the minimal polynomial $m_{\underline{x}}(X)$ of $\mathfrak{A}$ or to $m_{\underline{x}}(X)X$ depending to $\mathfrak{A}$ has no unit elements or has a unit element and in the former case $q_{\underline{x}}(X)$ divides the minimal polynomial $m_{L(\underline{x})}(X)$ of the linear transformation $L(\underline{x})$ of $\mathfrak{A}_F$ defined by $L(\underline{x}) := \underline{x}a$ for all $a \in \mathfrak{A}_F$(cf. [1], p.33).

By definition we have the equation

$$q_{\underline{x}}(\underline{x}) = \underline{x}^r + \beta_1(\underline{x})\underline{x}_{r-1} + \ldots + \beta_{r-1}(\underline{x})\underline{x} = 0.$$

Definition 2. A commutative algebra $\mathfrak{A}$ is baric if it admits a non trivial algebra homomorphism $\omega : \mathfrak{A} \to K$. The homomorphism ω is called the weight homomorphism.

From the algebra homomorphism theorem follows

Proposition 1 ([1], Theorem 1.8). Let $\mathfrak{A}$ be an n-dimensional baric algebra over K with weight homomorphism ω. Then $\mathfrak{N} := \ker\omega$ is an $(n-1)$-dimensional ideal of $\mathfrak{A}$ and the quotient algebra $\mathfrak{A}/\mathfrak{N}$ is isomorphic to the basic field K.

Definition 3. Let $\mathfrak{A}$ be a baric algebra with a weight homomorphism ω. $\mathfrak{A}$ is called a train algebra if the coefficients $\beta_i(\underline{x})$ of the rank polynomial (1) of $\mathfrak{A}$ are functions of $\omega(\underline{x})$ only.

In view of the above proposition, in a train algebra $\mathfrak{A}$ with a weight homomorphism ω the coefficients $\beta_i(\underline{x})$ are constant multiples of the i-th power $\omega^i(\underline{x})$ of the weight $\omega(\underline{x})$ and thus the rank equation of $\mathfrak{A}$ has the form

$$q_{\underline{x}}(\underline{x}) = \underline{x}^r + \gamma_1\omega(\underline{x})\underline{x}^{r-1} + \gamma_2\omega^2(\underline{x})\underline{x}_{r-2} + \ldots + \gamma_{r-1}\omega_{r-2}(\underline{x})\underline{x} = 0, \tag{2}$$

where the constants γ_i are in the field F. In a train algebra $\mathfrak{A}$ of rank r all elements $x \in \mathfrak{A}$ of weight 1 satisfy the equation

$$x^r + \gamma_1 x^{r-1} + \gamma_2 x^{r-2} + \ldots + \gamma_{r-1}x = 0. \tag{3}$$

In any extention field L of K which contains the splitting field of the polynomial

$$p(X) := X^r + \gamma_1 X^{r-1} + \gamma_2 X^{r-2} + \ldots + \gamma_{r-1}X,$$

$p(X)$ splits into linear factors

$$p(X) = X(X-\lambda_0)(X-\lambda_1)\ldots(X-\lambda_{r-2}), \tag{4}$$

where the nonassociative products of factors in the right-hand side can be accomplished one by one in certain direction, e.g. from left to right (cf.[1], pp.26−27). Without loss of generality we can assume that $\lambda_0 = 1$ as $p(1) = 0$.

Definition 4. Let $\mathfrak{A}$ be a train algebra with the rank equation (2) and let the polynomial $p(X)$ split into linear factors in $L \supseteqq K$ as in (4). Then the elements $\lambda_0(=1)$, $\lambda_1,\ldots,\lambda_{r-2}$ of L are called the principal train roots of $\mathfrak{A}$.

Then $q_{\underline{x}}(\underline{x})$ can be factorized as

$$q_{\underline{x}}(\underline{x})=\underline{x}(\underline{x}-\lambda_0\omega(\underline{x}))(\underline{x}-\lambda_1\omega(\underline{x}))\dots(\underline{x}-\lambda_{r-2}\omega(\underline{x})). \tag{5}$$

The sense of products in the right-hand side is same as the one in (4).

We remark that a train algebra has exactly one weight homomorphism (cf.[1], Corollary 3.6). From (2) immediately follows

Proposition 2. Let $\mathfrak{A}$ be a train algebra with the weight homomorphism $\omega:\mathfrak{A}\to K$. Then every element in $\mathfrak{N}$ = kerω is nilpotent of an index not greater than r.

Let $M(\mathfrak{A})$ be the algebra of all linear mappings of $\mathfrak{A}$ into itself.
$M(\mathfrak{A})$ is associative and isomorphic to the n^2-dimensional matrix algebra $M_n(K)$ over K. Let $T(\mathfrak{A})$ be the subalgebra of $M(\mathfrak{A})$ generated by the identity transformation I and all linear transformations $L(a)$ of $\mathfrak{A}$, so-called (left) transformations corresponding to a defined for each $a\in\mathfrak{A}$ by

$$L(a)x:=ax \text{ for all } x \text{ in } \mathfrak{A}.$$

(Generally speaking also right transformation $R(a)$ can be defined by $R(a)x:=xa$ but it coincides with $L(a)$ in the commutative case.)
The algebra $T(\mathfrak{A})$ is called the transformation algebra of $\mathfrak{A}$. Every transformation $T\in T(\mathfrak{A})$ can be represented in the form

$$T=f(L(a_1),\dots,L(a_s)),$$

where $f(L(a_1),\dots,L(a_s))$ denotes a linear combination of the identity transformation and products of $L(a_1),\dots,L(a_s)$ corresponding to certain $a_1,\dots,a_s\in\mathfrak{A}$, i.e. f is a polynomial of s associative, non-commutative indeterminates over K.

Proposition 3 ([1], Theorem 3.13). Let $\mathfrak{A}$ be an n-dimensional commutative algebra over K. Then the following conditions are equivalent:

(A) $\mathfrak{A}$ is a baric algebra with a weight homomorphism ω such that the coefficients of the characteristic polynomial of any transformation $T=f(L(a_1),\dots,L(a_s))\in T(\mathfrak{A})$ are functions of $\omega(a_1),\dots,\omega(a_s)$ only.

(B) The extention algebra $\mathfrak{A}_L$ of $\mathfrak{A}$, where L is an algebraic extension of K, admits a basis $c_0, c_1,\dots,c_m$, $c_0\in\mathfrak{A}$, $m:=n-1$, such that the multiplication constants $\lambda_{ijk}\in L$ defined by

$$c_ic_j=\sum_{k=0}^{m}\lambda_{ijk}c_k,\quad i,j=0,\dots,m,$$

have the following properties

(i) $\lambda_{000}=1$,

(ii) $\lambda_{0jk}=0$ for $0\leqq k<j\leqq m$,

(iii) $\lambda_{ijk}=0$ for $0\leqq k\leqq\max(i,j)$ if $i,j>0$.

Definition 5. A commutative algebra $\mathfrak{A}$ which satisfies one of the conditions (A), (B) stated above is called genetic. Such a basis $c_0, c_1,\dots,c_m$ of $\mathfrak{A}_L$ as characterized in (B) ia called a canonical basis of $\mathfrak{A}$ over L.

The multiplication constants $\lambda_{000}(=1)$, $\lambda_{011},\dots,\lambda_{0mm}$ with respect to a canonical basis $c_0,c_1,\dots,c_m$ of an $(m+1)$-dimensional genetic algebra $\mathfrak{A}$ over L are uniquely determined since it is shown that they are the characteristic roots of any transformation $\tilde{L}(x)$ of $\mathfrak{A}_L$ which is a natural extention of a transformation $L(x)$ of $\mathfrak{A}$ corresponding to $x\in\mathfrak{A}$ with $\omega(x)=1$.

Definition 6. Let $\mathfrak{A}$ be a genetic algebra with a canonical basis $c_0, c_1,\dots,c_m$ over $L\supseteqq K$ and the multiplication constants λ_{ijk}, $i,j,k=0, 1,\dots,m$. Then $\lambda_{000}(=1),\lambda_{011},\dots,\lambda_{0mm}$ are called the train roots of $\mathfrak{A}$.

Proposition 4 ([1], Theorem 3.10). Let $\mathfrak{A}$ be baric algebra. Then if $\mathfrak{A}$ is genetic, it is a train algebra.

The principal train roots of genetic algebra $\mathfrak{A}$ are contained in the train roots of $\mathfrak{A}$.
The finite-dimensional genetic algebras form a proper subclass of the finite-dimensional train algebras (cf.[1], Theorem 3.21).

It follows from Proposition 4 and Proposition 2 that if $\mathfrak{A}$ is baric with a weight homomorphism ω and is genetic, $\mathfrak{N}=\ker\omega$ is a nilpotent ideal of $\mathfrak{A}$. The principal powers $\mathfrak{N}^r$ of $\mathfrak{N}$, $r\in N$, form a sequence of subalgebras of $\mathfrak{A}$, but are not necessarily ideals of $\mathfrak{A}$.

Definition 7. A baric algebra $\mathfrak{A}$ with a weight homomorphism ω is called a special train algebra if $\mathfrak{N}:=\ker\omega$ is nilpotent and the principal powers $\mathfrak{N}^r$ of $\mathfrak{N}$, $r\in N$, are ideals of $\mathfrak{A}$.

Proposition 6([1], Theorem 3.28). Let $\mathfrak{A}$ be a baric algebra. Then if $\mathfrak{A}$ is a special train algebra, it is genetic.

The finite-dimensional train algebras form a proper subclass of the finite-dimensional special train algebras(cf.[1], Theorem 3.30).

2. Decomposition of a train algebra.

We consider the decomposition of a train algebra $\mathfrak{A}$ into the vector space direct sum of the eigen spaces belonging to the transformation $L(e)$ of $\mathfrak{A}$ corresponding to an idempotent element e of $\mathfrak{A}$.

Let $\mathfrak{A}$ be a train algebra of rank r over a field K with weight homomorphism ω and let $\mathfrak{N}:=\mathrm{Ker}\omega$. We assume that $\mathfrak{A}$ has an idempotent element e, i.e. an element $e\neq 0$ with $e^2=e$ and that K is a field of characteristic $\neq 2$ with at least r elements and contains the splitting field of the characteristic polynomial of $L(e)$. We denote the restriction of $L(e)$ to $\mathfrak{N}$ by $\hat{L}(e)$.

Now let

$$q_{\underline{x}}(\underline{x})=\underline{x}^r+\gamma_1\omega(\underline{x})\underline{x}^{r-1}+\gamma_2\omega^2(\underline{x})\underline{x}^{r-2}+\dots+\gamma_{r-1}\omega^{r-2}(\underline{x})\underline{x}=0 \quad (6)$$

be the rank equation of $\mathfrak{A}$ and let $\gamma_0(=1)$, $\lambda_1,\dots,\lambda_{r-2}$ be the principal train roots of $\mathfrak{A}$. Then by definition all elements x of $\mathfrak{A}$ satisfy

$$q_{\underline{x}}(x)=x(x-\omega(x))(x-\lambda_1\omega(x))\dots(x-\lambda_{r-2}\omega(x))=0. \quad (7)$$

Since ω is an algebra homomorphism, we can use the linearization process(cf. Osborn [2]) by acting the operators $\delta_x{}^i(y)$, $i=1,2,\dots$, where $\delta_x{}^i(y)$ has the effect of replacing each monomial in

$$q_{\underline{x}}(x)=x^r+\gamma_1\omega(x)x^{r-1}+\dots+\gamma_{r-1}\omega^{r-2}(x)x$$

by the sum of all monomials that can be obtained from it by replacing i of x in the monomial by $y\in\mathfrak{A}$. Then we can compute the resultant by using the familiar relations between the constants γ_j, $j=1,\dots,r-1$, in (6) and λ_k, $k=0,\dots,r$, in (7). For example acting $\delta_x{}^1(y)$ on $x^4+\gamma_1\omega(x)x^3+\gamma_2\omega^2(x)x^2+\gamma_3\omega^3(x)x=0$(the case $r=4$ for (6)) yields $[2x(x(xy))+x(x^2y)+x^3y]-(1+\gamma_1+\gamma_2)[\omega(x)\{2x(xy)+x^2y\}+\omega(y)x^3]+2(\lambda_1+\lambda_2+\lambda_1\lambda_2)[\omega^2(x)xy+\omega(x)\omega(y)x^2]-\lambda_1\lambda_2[\omega^3(x)y+3\omega(x)^2\omega(y)x]=0$. In the expression above putting $x=e$, $y\in\mathfrak{N}$,i.e. $\omega(y)=0$, then we obtain

$$(L(e)-\lambda_1I)(L(e)-\lambda_2I)(L(e)-\frac{1}{2}I)y=0 \quad \text{for all } y\in\mathfrak{N}.$$

Similarly in general we can obtain

$$(L(e)-\lambda_1I)\dots(L(e)-\lambda_{r-2}I)(L(e)-\frac{1}{2}I)y=0 \quad \text{for all } y\in\mathfrak{N}. \quad (8)$$

$$\text{or}\quad (\hat{L}(e)-\lambda_1I)\dots(\hat{L}(e)-\lambda_{r-2}I)(\hat{L}(e)-\frac{1}{2}I)=0. \quad (9)$$

From(8) follows

$$\lambda_1\lambda_2\dots\lambda_{r-2}y=0 \text{ if } ey=0.$$

Also from (9) follows that the possible characteristic roots of the transformation $\hat{L}$(e) of $\mathfrak{N}$ are $\lambda_1,\lambda_2,\dots,\lambda_{r-2}$ or 1/2 and then by means of Osborn [2], Corollary 6.3, we have

Theorem A. Let $\mathfrak{A}$ be a train algebra of rank $r\geqq 3$ over a field K of characteristic $\neq 2$ with at least r elements. Suppose that $\mathfrak{A}$ has an idempotent element e and that K includes the splitting field of the characteristic polynomial of the transformation $L(e)$. Let $\mathfrak{N}$ be the kernel of the

weight homomorphism ω of $\mathfrak{A}$ and let $\lambda_0(=1)$, $\lambda_1,\ldots,\lambda_{r-2}$ be the principal train roots of $\mathfrak{A}$.
If none of $\lambda_1,\ldots,\lambda_{r-2}$ equal 1/2, then

$$\mathfrak{A}=\mathfrak{G}\oplus\mathfrak{N}(\lambda_1)\oplus\ldots\oplus\mathfrak{N}(\lambda_{r-2})\oplus\mathfrak{N}(1/2),$$

where $\mathfrak{G}:=\langle e\rangle$, $\mathfrak{N}(\lambda_i):=\{y\in\mathfrak{N}\mid(L(e)-\lambda_i I)y=0\}, i=1,\ldots,r-1$ and $\mathfrak{N}(1/2):=\{y\in\mathfrak{N}\mid(L(e)-1/2I)y=0\}$.

Remark. It is shown by Wörtz-Busekros[1], Theorem 9.12, that a train algebra $\mathfrak{A}$ over K of characteristic $\neq 2$ of rank 3 with principal train roots $\lambda_0=1$, $\lambda_1=0$ is a Bernstein algebra that admits the vector space decomposition $\mathfrak{A}=\mathfrak{N}(0)\oplus\mathfrak{N}(1/2)$ according to the above notation.
Moreover such a Bernstain algebra is a Jordan algebra provided characteristic of K different from 2,3,5 (cf. Miyamoto[3], Theorem 5). From this point of view Theorem A seems to be a generalization of Wörz-Busekros's theorem to case of $r\geqq 3$.

3. Characterization of special train algebras

Let $\mathfrak{A}$ be a baric algebra with a weight homomorphism ω and let $T(\mathfrak{A})$ be the transformation algebra of $\mathfrak{A}$. Then the Lie algebra $T(\mathfrak{A})^-$ associated with $T(\mathfrak{A})$ is obtained by defining a new multiplication $[x,y]:=xy-yx$ in the same vector space as $T(\mathfrak{A})$.
In 1972 Holgate, P. has given a characterization of genetic algebras. His result is restated in Wörz-Busekros[1], Theorem 3.22, in a slightly weakened form as following

Theorem. Let $\mathfrak{A}$ be a baric algebra with a weight homomorphism ω over an algebraic closed field K of characteristic 0 and let $T(\mathfrak{A})^-$ be the Lie algebra associated with the transformation algebra $T(\mathfrak{A})$ of $\mathfrak{A}$. Then the following statesments are equivalent:
(i) $\mathfrak{A}$ is genetic.
(ii) $T(\mathfrak{A})^-$ is solvable and $\mathfrak{N}:=\mathrm{Ker}\,\omega$ is nilpotent.

We will give a characterization of special train algebras that corresponds to one of train algebras stated in the above theorem, that is

Theorem B. Let $\mathfrak{A}$ be a finite-dimensional baric algebra with a weight homomorphism ω over an algebraic closed field K of characteristic 0 and let $T(\mathfrak{A})^-$ be the Lie algebra associated with the transformation algebra $T(\mathfrak{A})$ of $\mathfrak{A}$. Then the following statements are equivalent:
(i) $\mathfrak{A}$ is a special train algebra.
(ii) $T(\mathfrak{A})^-$ is nilpotent and $\mathfrak{N}:=\mathrm{Ker}\,\omega$ is nilpotent.

Proof. As usual we identify each element of $T(\mathfrak{A})^-$ with its corresponding matrix respectively with respect to a fixed basis of $\mathfrak{A}$.

[(i)$\Rightarrow$(ii)]. Let $\mathfrak{A}$ be a special train algebra. Then by Definition 7(§1) $\mathfrak{N}$ is nilpotent and the principal powers $\mathfrak{N}^r$ of $\mathfrak{N}$, $r\in N$, form a decending chain of ideals in $\mathfrak{A}$ such that $\mathfrak{N}\supseteqq\mathfrak{N}^2\supseteqq\ldots\supseteqq\mathfrak{N}^r\supseteqq\mathfrak{N}^{r+1}=\{0\}$ for a suitable integer r. Then according to Wörz-Busekros [1], pp.56-58, we consider matrix representations of elements of $T(\mathfrak{A})^-$ with respect to a basis of $\mathfrak{A}$ chosen as the following:

Let an element c_0 of $\mathfrak{A}$ with $\omega(c_0)=1$ be fixed and choose a basis $c_1{}^{(1)},\ldots,c_{k_1}{}^{(1)},\ldots,c_1{}^{(r)},\ldots,c_{kr}{}^{(r)}$ of $\mathfrak{N}$ such that $c_1{}^{(l)},\ldots,c_{kl}{}^{(l)}\in\mathfrak{N}^l-\mathfrak{N}^{l+1}$ for $l=1,2,\ldots,r-1$, where $k_\ell:=\dim\mathfrak{N}^l-\dim\mathfrak{N}^{l+1}$. Then with respect to a basis c_0, $c_1{}^{(1)},\ldots,c_{k_1}{}^{(1)},\ldots,c_1{}^{(r)},\ldots,c_{kr}{}^{(r)}$ of $\mathfrak{A}$ the transformations corresponding to base elements are represented simultaneously by matrices partitioned into blocks as following
$L(c_i{}^{(l)})=$

$$\begin{pmatrix}
B_{00} & B_{01} & \cdots & B_{0r}\\
B_{10} & B_{11} & \cdots & B_{1r}\\
\vdots & \vdots & & \vdots\\
B_{l0} & B_{l1} & \cdots & B_{lr}\\
\vdots & \vdots & & \vdots\\
B_{r0} & B_{r1} & \cdots & B_{rr}
\end{pmatrix}$$

for fixed l, $i(1\leqq l\leqq r,\ 1\leqq i\leqq k_l)$, where the type of each block is respectively, B_{00}:1×1, B_{s0}:$k_s\times 1$, B_{ot}: $1\times k_t$, B_{st}:$k_s\times k_t$ for $s,t=1,2,\ldots,r$, and B_{pq} are zero matrices for all $p=0,\ldots,l-1$, $q=0,\ldots,r$ and for all p, q such that $l\leqq p\leqq q\leqq r$ and

$$L(c_0)=\left(\begin{array}{c:c:c:c} C_{00} & C_{01} & \cdots & C_{0r} \\ \hdashline C_{10} & C_{11} & \cdots & C_{1r} \\ \hdashline \vdots & \vdots & & \vdots \\ \hdashline C_{l0} & C_{l1} & \cdots & C_{lr} \\ \hdashline \vdots & \vdots & & \vdots \\ \hdashline C_{r0} & C_{r1} & \cdots & C_{rr} \end{array}\right)$$

where the type of each block C_{pq} corresponds to the one of B_{pq} respectively, and C_{pq} are zero matrices for all p,q such that $0\leqq p<q<r$ and $C_{00}=1$.

Since K is algebraicaly closed, we can assume that $c_0,c_1{}^{(1)},\ldots,c_{k1}{}^{(1)},\ldots,c_1{}^{(r)},\ldots,c_{kr}{}^{(r)}$ are chosen so that with respect to the base

$$c_1{}^{(l)}+\mathfrak{N}^{l+1},\ldots,c_{kl}{}^{(1)}+\mathfrak{N}^{l+1}$$

of $\mathfrak{N}^l/\mathfrak{N}^{l+1}$, $l=1,\ldots,r$, the linear transformation $L(c_0)_\ell$ induced by $L(c_0)$ in $\mathfrak{N}^l/\mathfrak{N}^{l+1}$ are represented by lower triangular matrices of the form

$$\left(\begin{array}{|c|c|c|c|} \hline \begin{matrix}\alpha & & \\ & \ddots & \\ & & \alpha\end{matrix} & O & \cdots & O \\ \hline * & \begin{matrix}\beta & & \\ & \ddots & \\ & & \beta\end{matrix} & \cdots & O \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline * & * & \cdots & \begin{matrix}\gamma & & \\ & \ddots & \\ & & \gamma\end{matrix} \\ \hline \end{array}\right), \tag{10}$$

where $\alpha,\beta,\ldots,\gamma$ are the characteristic roots of $L(c_0)_l$, $l=1,\ldots,r$.

Then with respect to a basis c_0, $c_1{}^{(1)},\ldots$, $c_{k1}{}^{(1)},\ldots$, $c_1{}^{(r)},\ldots$, $c_{kr}{}^{(r)}$ $L(c_i{}^{(\ell)})$, $i=1,\ldots,k_l$, $l=1,\ldots,r$, are represented by nilpotent lower block triangular matrices and $L(c_0)$ is represented by a lower block triangular matrix such that its diagonal blocks have the form (10).

So any element of $T(\mathfrak{A})^-$ is represented by a lower block triangular matrix such that its diagonal blocks have the same form as (10). Therefore $T(\mathfrak{A})^-$ is nilpotent.

[(ii)$\Rightarrow$(i)]. We assume that $T(\mathfrak{A})^-$ is nilpotent and $\mathfrak{N}$ is nilpotent.

Let c_0 be an element of $\mathfrak{A}$ with $\omega(c_0)=1$. Then $\mathfrak{A}=\langle c_0\rangle\oplus\mathfrak{N}$.

Now let $T_0(\mathfrak{A})$ be the subalgebra of $M(\mathfrak{N})$, the algebra of all linear mappings of $\mathfrak{N}$ into itself, which consists of the restrictions $\hat{T}$ of all $T\in T(\mathfrak{A})$ to $\mathfrak{N}$. Since $T(\mathfrak{A})^-$ is nilpotent, $T_0(\mathfrak{A})^-$, i.e. the Lie algebra associated with $T_0(\mathfrak{A})$, also is a split nilpotent Lie algebra of linear transformations of $\mathfrak{N}$. Hence we can apply the well-known theorem(cf. Jacobson [1], Theorem 12) obtained by the combination of Lie's theorem with the weight-space decomposition theorem for a split nilpotent Lie algebra of linear transformations in a finite-dimensional vector space over a field of characteristic 0. The result is as following:

$\mathfrak{N}=\oplus_\Sigma\mathfrak{N}_{\sigma j}$, direct sum of weight spaces $\mathfrak{N}_{\sigma j}$,
Σ: the set of weights of $\mathfrak{N}$ for $T_0(\mathfrak{A})^-$, and

there exists a basis of $\mathfrak{N}$

$$a_1,\ldots,\ a_{i_1},a_{i_1+i_2},\ldots,a_{i_1}i,\ldots,a_{i_1+\ldots+i_{s-1}}\ldots,\ a_{i_1+\ldots+i_s},$$

such that $a_{i_1+\ldots+i_{j-1}+1},\ \ldots,\ a_{i_1+\ldots+i_{j-1}+i_j}$ are base of $\mathfrak{N}_{\sigma j}$, $j=1,\ \ldots,\ s$, $s:=\mathrm{Card}(\Sigma)$, $i_j:=\dim(\mathfrak{N}_{\sigma j})$ with respect to which transformations $\hat{T}\in T_0(\mathfrak{A})^-$ in the weight space $\mathfrak{N}_{\sigma j}$ can be represented simultaneously by lower triangular matrices with $\sigma_j(\hat{T})$ as diagonal components.

Let α_{pqr} be multiplication constants of $\mathfrak{N}$ defined by

$$a_p a_q=\sum_r\alpha_{pqr}a_r,\ \ (1\leqq r\leqq t:=i_1+\ldots+i_{s-1}+i_s).$$

Then since $\mathfrak{N}_{\sigma j}$ are $T_0(\mathfrak{A})^-$-submodules,

(i) If $a_p\in\mathfrak{N}_{\sigma j}$, $a_q\in\mathfrak{N}_{\sigma k}$, $j\neq k$, then $\alpha_{pqr}=0$ for $r=1,\ldots,t$.

(ii) If a_p, $a_q \in \mathfrak{N}_{\sigma j}$ and $i_1+\ldots+i_{j-1}+1 \leqq r \leqq i_1+\ldots+i_{j-1}+i_j$, then $\alpha_{pqr}=0$ for all $r<q$ and $\alpha_{pqr}=\sigma_j(\hat{L}(a_p))$ for $r=q$.

Moreover for nilpotency of $\mathfrak{N}$ follows that of $\hat{L}(a_p)$ (cf. Schafer[5], Theorem 2.4), hence $\sigma_j(\hat{L}(a_p)) =0$, i.e.

(iii) If a_p, $a_q \in \mathfrak{N}_{\sigma j}$, then $\alpha_{pqr}=0$.

According to (i), (ii), (iii) we obtain

(iv) $a_p a_q \in \langle a_{q+1},\ldots,a_{i_1+\ldots+i_j}\rangle$ if $a_q \in \mathfrak{N}_{\sigma j}$,

from which directly follows that the principal powers $\mathfrak{N}^r$ of $\mathfrak{N}$, $r \in N$, are ideals in $\mathfrak{A}$. Thus by definition $\mathfrak{A}$ is a special train algebra. Q.E.D.

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